TELEMETRY AND JUGGLING

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ABSTRACT

One of the beauties of mathematics is its ability to demonstrate the relationship between apparently unrelated subjects. And this is not only an aesthetic attribute. The insight obtained by seeing relations where they are not obvious often leads to elegant solutions to difficult problems. This paper will demonstrate a mathematical relation between telemetry and juggling. Any given pulse code modulation (PCM) format can be mapped onto a juggling pattern. The Inter-Range Instrumentation Group (IRIG) 106 Class I PCM formats are a subset of all juggling patterns while the Class II PCM formats are equivalent to the set of all juggling patterns (within some mathematically precise definitions). There are actually quite a few mathematical results regarding juggling patterns. This paper will also discuss how these topics relate to tessellations, bin packing, PCM format design, and dynamic spectrum allocation. One of the shortcomings of human nature is the tendency to get caught up in a particular topic or viewpoint. This is true of the telemetry community as well. It is hoped that this paper will increase the awareness that there are a variety of areas of theory outside of telemetry that may be applicable to the field.

INTRODUCTION

This is the tale of two seemingly unrelated activities: the periodic sampling of data and the continuous tossing of objects into the air. As we shall see, these two activities, which we refer to as telemetry and juggling, actually have a common point of intersection. This will be demonstrated through a mathematical description of both. More specifically, it will be shown that the patterns through time described by sampling and by tossing are equivalent.

Once this has been done, a brief but broad foray will be taken into related topics. A common thread will be sewn through areas as diverse as data cycle map design and tessellations, spectrum assignment and dynamic bin packing, as well as scheduling and memory mapping. When all is done, it is hoped that the reader will have been exposed to the beauty of mathematical associations and that the reader will have found the nugget of a solution to a problem of their own.

TELEMETRY

A common objective during a test is to collect periodic samples of many measurands (e.g., temperature, pressure, etc.). In order to monitor these measurands in real time, they are commonly telemetered using a data cycle map (DCM). [DCMs are often specified using a pulse code modulated (PCM) matrix.] Before designing a DCM, the measurands and their required sampling rates are specified. For the purposes of this paper, we are only interested in the set of sample rates used in a DCM. Thus, we will consider a set of sample rates, $S = \{s_i\}$, to be the input to a DCM design. For example, the set $S = \{4,12,2,2,1,1,1,1\}$ indicates one measurand at a rate of 4 samples per second (sps), one measurand at 12 sps, two measurands at 2 sps, and four measurands at 1 sps. These might be arranged in a DCM as shown in Figure 1, where the number in each cell corresponds to the subscript i in the set $\{s_i\}$. In the example, the number 2 shows up 12 times in the DCM because it represents s_2. For simplicity we assume one frame of a DCM is sent in one second. Each cell in this matrix represents a word consisting of a set of bits. Thus, the DCM represents the cyclical definition of a bit stream where the bits are sent row by row.

1	2	3	2	5	2
1	2	4	2	6	2
1	2	3	2	7	2
1	2	4	2	8	2

Figure 1 – A DCM.

JUGGLING

A common objective during juggling is to throw many objects into the air in various patterns (without dropping them). The standard three-ball cascade is illustrated in Figure 2. Each ball is thrown in a figure eight pattern alternating from hand to hand. The standard three-ball shower, shown in Figure 3, throws one ball high from one hand to the other, then throws the same ball short across to the original hand.

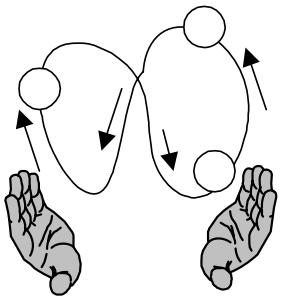

Figure 2
Three-Ball Cascade

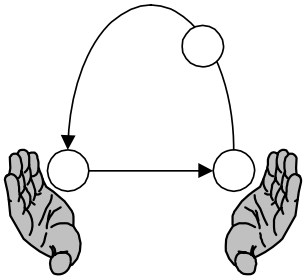

Figure 3
Three-Ball Shower

There is a notation for describing juggling patterns referred to as *site swap* [1]. This notation models juggling by considering every throw of a ball as a discrete point in time. Each throw is described by an integer that indicates how many points in time after throwing a ball, the ball is caught and thrown again. This can be illustrated using a series of points and drawing arcs between points to indicate the path of a ball through time. The three-ball cascade is described using site swap notation by a "3". This is because every ball thrown comes down 3 time slots later as illustrated in Figure 4. The

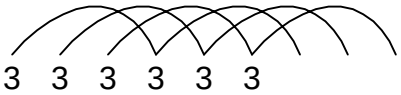

Figure 4
Three-Ball Cascade
Site Swap Notation

notation for the three-ball shower is “5 1” (read “five one”), because each ball is thrown such that it comes down 5 time slots later, is thrown again, and comes down one time slot later. (Repeat.) This is shown in Figure 5. When trying to map site swap notation to normal, single person, two handed, juggling, it is good to remember that throws are done by alternating hands. Thus, you might add a series of alternating L’s and R’s (for left hand, right hand) under the time slots in Figures 4 and 5. In the two handed model, this means that throws of odd numbers change hands and throws of even numbers return to the same hand. The standard four-ball cascade (a “4” in site swap) is actually two balls in each hand at the same time.

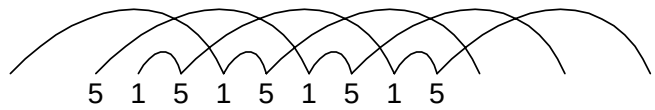

Figure 5 - Three-Ball Shower Site Swap Notation

As can be imagined, the more time slots later a ball comes down, the higher the ball has to be thrown. This is why the numbers in site swap notation are referred to as the *height function* of the throw. (Although, if you recall a little physics, the height increases more closely to the square of the time in the air.) The site swap notation for a juggling pattern can thus be represented by a set $H = \{h_i\}$, $i = 1,\ldots,F$, where each h_i represents the height function of a throw, and F is the number of throws in the pattern.

There are actually quite a few academic publications regarding juggling [2]. Some of the mathematical results regarding the nature of juggling patterns will be applied to DCMs. In particular it has been shown [3] that a sequence of integers $\{h_i\}$ represents a juggling pattern if and only if the following two conditions are met:

(1) No two balls are caught or thrown at the same time.

(2) The average of the height functions, $\frac{\sum h_i}{F}$, is an integer (and the number of balls).

This characterization will be used to demonstrate the equivalence of telemetry and juggling.

EQUIVALENCE

On an informal level, mapping measurands to balls shows the equivalence between DCMs and juggling. That is, each throw of a ball is equivalent to sampling a measurand. The following formal demonstration transforms the sampling rate set S, into the height function set H. To do this, some notation needs to be defined.

A DCM is defined using a sequence of integers $\{m_{i,j}\}, i = 1,\ldots,M,\ j = 1,\ldots,s_i$, where M is the number of measurands and s_i is the sample rate for measurand i. Each $m_{i,j}$ represents a sample of measurand i. Set $m_{i,j} = i$ for each i and all j. This is exactly what was done in the example DCM in Figure 1. (Note that the subscripts, i and j, do not represent the rows and columns of the matrix. To define the cell for each $m_{i,j}$ would require a mapping function from $\{m_{i,j}\}$ onto the cells of the matrix. This is not needed for the current proof.) In the DCM shown in Figure 1, $\{m_{i,j}\}$ = {1,2,3,2,5,2,1,2,4,2,6,2,1,2,3,2,7,2,1,2,4,2,8,2}.

Further, define the *frame size*, $F = \sum s_i$, to be the total number of samples, or the total number of words, in the DCM. In our example, $F = 24$. Finally, define $h(m_{i,j}) = \frac{F}{s_i}$ to be the height function for the samples with sample rate s_i. Thus, in our example, the samples for a measurand with sample rate 4 have heights of $\frac{24}{4} = 6$. The full set of height functions in our example is $\{h(m_{ij})\}$ = {6,2,12,2,24,2,6,2,12,2,24,2,6,2,12,2,24,2,6,2,12,2,24,2}. This set is also equal to H, the set of site swap height functions.

<u>*Theorem*</u>. Every DCM can be associated with a juggling pattern.

<u>*Proof*</u>. By construction, no two samples in a DCM can be transmitted at the same time. This is equivalent to condition (1) that no two balls are caught or thrown at the same time. That condition (2) is met by a DCM is shown by

$$\frac{\sum_{i=1}^{M}\sum_{j=1}^{s_i} h(m_{i,j})}{F} = \frac{\sum_{i=1}^{M}\frac{F}{s_i}s_i}{F} = \frac{F\sum_{i=1}^{M}1}{F} = M = \text{number of measurands} = \text{number of balls.}$$

Q.E.D

This shows that every DCM is a juggling pattern. The converse is not true. The {5,1} pattern can not be associated with a telemetry map since the balls (samples) do not have a constant height function. There is a caveat, however. The way we have defined DCMs restricts them to IRIG 106 [4] Class I DCMs, which require periodic sampling. Class II DCMs do not require periodic sampling. The removal of the periodic requirement makes the set of DCMs exactly the same as the set of juggling patterns.

This provides a nice mathematical equivalence between the two activities. In practice, however, there are some very distinct differences. The current record for most objects juggled is 12 rings (although there are rumors that 14 will be demonstrated soon). In flight test, requirements are getting into the 10s of thousands of measurands to be telemetered. Thus, at first glance, it appears that there is little to be learned from this relation and, perhaps, there is not a lot to be learned here, but there are at least a few nuggets to be gleaned.

The process of mapping $\{m_{i,j}\}$ onto a DCM is exactly the DCM design problem. This is known to be a mathematically complex problem (technically it is NP-Complete) [5]. Theorem 3 in [3] gives some insight into why this is true. Specifically, the number of period n juggling patterns with fewer than b balls is b^n. Or, in terms of telemetry, the number of DCMs with frame size F and m measurands is F^m. So, a full search of all DCMs for a given frame size requires an exponential growth in search time as the number of measurands increases.

A solution to the DCM design problem is important in that, every time a test involving telemetry is to be executed, a DCM must be designed. There are several projects being implemented that have come up

with efficient algorithms for finding near optimal solutions to the DCM design problem, but better solutions could be useful. A mathematically interesting question that could shed light on this design problem is:

> Characterize sample rate sets that can be realized as a DCM of minimum frame size.

The two conditions that describe when a sequence of integers is a height function sequence for a juggling pattern is an elegant characterization. Part of this characterization's elegance, is that it leads to a linear algorithm for determining if a sequence of numbers represents a juggling pattern. In contrast, there is no fast algorithm for determining if a set of sample rates leads to a full DCM. The difference in difficulty is that a juggling pattern described by site swap has the height functions already sequenced. Whereas, by starting with the sample rates, the sequence of the samples, $\{m_{ij}\}$, must be determined. A juggling version of this problem is: Given a set of throw heights, find a permutation that generates a juggling pattern if such a permutation exists.

Another question that, if answered, could shed light on the DCM design problem is:

> What is the minimum DCM frame size that will accommodate a set of sample rates?

This question does not assume that all samples of a measurand will fit into a DCM such that no words in the DCM are unused.

RELATED TOPICS

In this section, we take a whirlwind tour through various areas of discrete mathematics, computer science, and test and evaluation.

One way of looking at the DCM design problem is as an integer tiling problem. If the words in a DCM are represented by integers, then the samples for a given measurand can be represented by a set of integers. These sets can be considered integer tiles. The DCM design problem is thus to find a tiling of all such tiles associated with a test. That is, find placements for each of the tiles (measurand samples) on the integers (DCM) such that no two tiles intersect (no two measurands are sampled at the same time). Such a tiling solution is a tessellation of the integers in the same way that many of M. C. Ecsher's works are tessellations of the plane.

Some tessellation problems are known as bin packing problems. That is, given a set of objects and a set of bins, how do you place the objects in the bins? One application of a particular bin packing algorithm is used by the Unix operating system to try to minimize memory allocation during run time. This same algorithm has been used to try to take an existing DCM and optimize it by rearranging the allocation of the samples in their bins. This is actually a dynamic form of bin packing, meaning that the objects in the bins come and go and the algorithm tries to minimize fragmentation over time. Dynamic bin packing algorithms are being applied to spectrum assignment. Within the context of shrinking telemetry spectrum and growing spectrum demand, this is an important process to optimize. The objects in this case are frequency bandwidths and the bins are portions of the spectrum. But this leads into the more general problem of scheduling. The operational process of running a test often requires coordinating a

vast array of resources such as aircraft, frequency, supplies, support, ranges, etc. This can also be thought of in terms of tessellations or bin packings – albeit multidimensional ones.

Many of these problems fit into the more general categories of discrete mathematics or computational geometry. Often these problems are computationally complex, so there is no efficient algorithm that provides an optimal solution. However, mathematicians and computer scientists have been studying these class of problems for decades and there are a variety of known algorithms for providing near optimal solutions in acceptable times. The trick to using these algorithms is to first, cast a problem into its mathematical equivalent and second, find the right researchers to tweek these algorithms to fit the particular problem.

CONCLUSION

It is not clear in my memory whether I realized the equivalence of juggling patterns and DCMs before I recognized the geometric nature of the DCM design problem. But it is clear to me that the recognition of this equivalence clarified, in my own mind, that the mathematical translation of the DCM problem I had made was valid and worth pursuing. Which it has, indeed, turned out to be. It is both the theoretical and practical beauty of this connection that I have tried to convey in this paper.

I have also hoped to try to broaden people's awareness of some areas of mathematics that are not well known; especially in terms of their practical applications. Just as there are many real world problems in search of solutions, there are many theoretical solutions in search of problems. The great conundrum of the industrial-academic interface is how to bring these two viewpoints together. If you have a difficult problem, perhaps a recasting of the problem will aid you in reaching a solution.

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